Combinations of Sequential and Simultaneous Games

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Motivation

Many real-world strategic situations combine both sequential and simultaneous decision-making.

Example 1: A firm enters a market (sequential), and competing firms then simultaneously set their prices.
Example 2: In international negotiations, one country may make a move first, followed by others responding simultaneously.

To analyze such games, we will use both game trees and normal form representations, depending on the structure of the game.

We will rely on the tools introduced so far—**rollback** for sequential parts and **best response analysis** (underlining) for simultaneous choices. (It will be a little more intricate than this.)

A Two-Stage Entry and Pricing Game

Consider two firms evaluating entry and pricing into a new market:

Stage 1: Entry Decision (Simultaneous)

- Each firm chooses whether to **enter** the market or **stay out**.
- Entry requires a cost of 10. If not enter, payoff is 0.

Stage 2: Pricing Decision (Simultaneous)

- If only one firm enters:
  - High price yields 24 $\rightarrow$ profit $=$ **14**.
  - Low price yields 15 $\rightarrow$ profit $=$ **5**.
- If both enter:
  - Both choose high $\rightarrow$ each gets 12 $\rightarrow$ profit $=$ **2**.
  - Both choose low $\rightarrow$ each gets 7.5 $\rightarrow$ profit $=$ **-2.5**.
  - One chooses low, the other high $\rightarrow$ low-price firm gets 15 (profit $=$ **5**); high-price firm gets 0 (profit $=$ **-10**).

Step 1: Solving by "Rollback"

To solve the game, we apply **rollback reasoning**: Start by analyzing the second-stage outcomes (pricing), then use this information to evaluate the first-stage entry decisions.

We consider three possible entry scenarios:

1. Only **Firm 1** enters.
2. Only **Firm 2** enters.
3. **Both firms** enter.

Each leads to different pricing games in the second stage. We solve each of these subgames to understand the payoffs from entering.

Second Stage: Only Firm 1 Enters

If only **Firm 1** enters, it acts as a monopolist and chooses its price:

- High price $\rightarrow$ Revenue $= 24$
- Low price $\rightarrow$ Revenue $= 15$

Firm 1 will optimally choose the **high price**, earning a revenue of **24**.

Conclusion: If only Firm 1 enters, its payoff is 24 (at that stage); Firm 2's payoff is 0.

The answer would be the same with exchanged payoffs if only Firm 2 enters.

Second Stage: Both Firms Enter

Now both firms enter and simultaneously choose prices. Payoffs depend on the price combination:

(High, High):	Each gets 12
(Low, Low):	Each gets 7.5
(Low, High):	Low-price firm gets 15
	High-price firm gets 0
(High, Low):	Same as above

Analogous to Prisoner's Dilemma: Each firm has a dominant strategy to choose **low price**, leading to (Low, Low) and negative profits.

Conclusion: If both enter, each firm earns -**2.5**.

Normal Form Games: Pricing and Entry Stages

Second Stage: Pricing Game (if both enter)

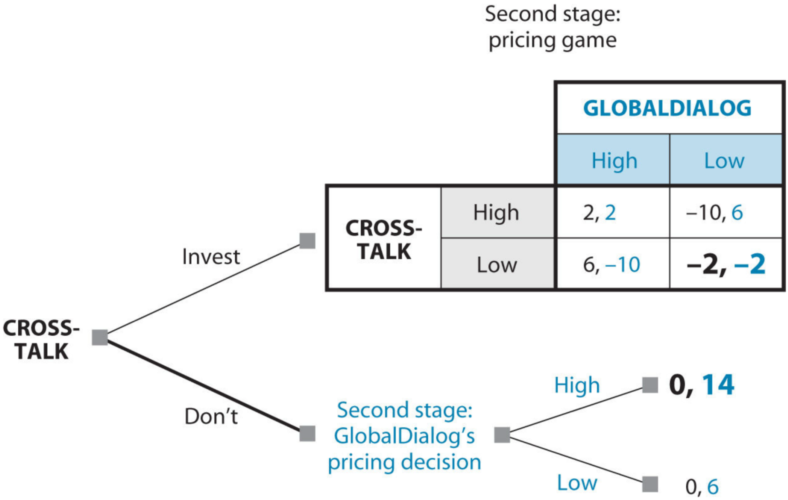
	High	Low
High	(12, 12)	(0, 15)
Low	(15, 0)	(7.5, 7.5)

First Stage: Entry Game (internalizing Investment)

	Enter	Stay Out
Enter	(-2.5, -2.5)	(14, 0)
Stay Out	(0, 14)	(0, 0)

We observe that there are two equilibria in which one firm enters and charges a high price.

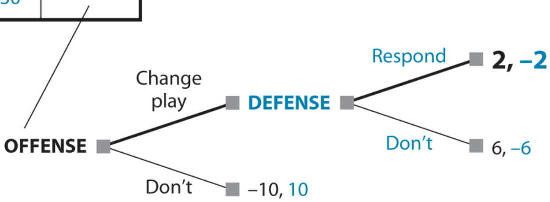
Other examples



Other examples

First stage:
coaches choose alignment

		DEFENSE TO COVER	
		Safe	Risky
OFFENSE TO PLAY	Safe	2, −2	6, −6
	Risky	30, −30	



Games with First-Mover Advantage

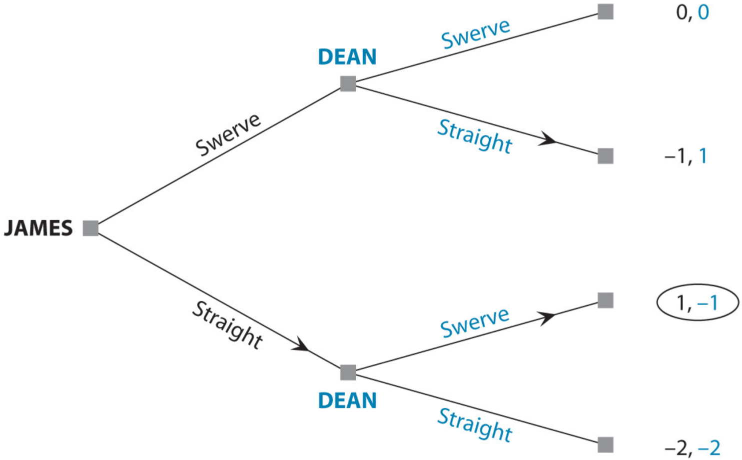
A game exhibits a **First-Mover Advantage** when the player who moves first receives a higher payoff in the rollback equilibrium than they would by moving second.

The comparison is not between the payoffs of Player 1 and Player 2, but rather between the payoffs a player receives when moving first versus when moving second in a similar role.

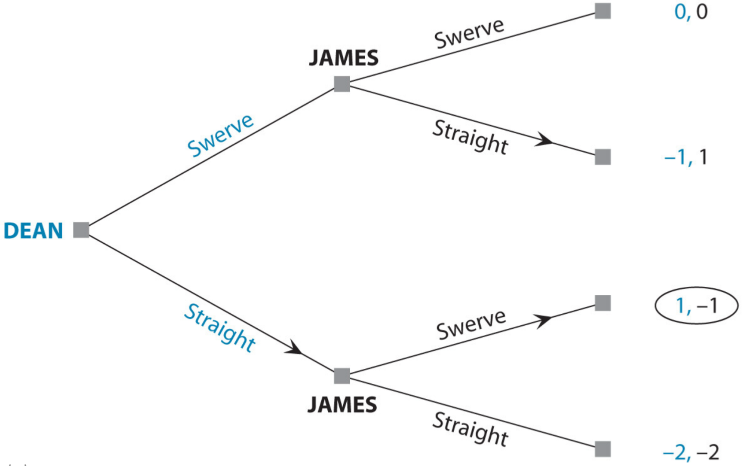
Rows represent James's strategies, and columns represent Dean's strategies.

	Swerve (Chicken)	Straight (Tough)
Swerve (Chicken)	(0, 0)	(-1, 1)
Straight (Tough)	(1, -1)	(-2, -2)

Games with First-Mover Advantage



Games with First-Mover Advantage



Games with Second-Mover Advantage

A game exhibits a **Second-Mover Advantage** when the player who moves second can observe the first player's choice and respond in a way that leads to a higher payoff than if they had moved first. This is the classic **Matching Pennies** game. Player 1 chooses rows, Player 2 chooses columns.

Payoffs:

	H	T
H	(1, -1)	(-1, 1)
T	(-1, 1)	(1, -1)

In the simultaneous game, there is no pure strategy equilibrium. But in a **sequential version**, whoever moves **second** can always observe and mismatch the first move — guaranteeing a win. Thus, there is a **second-mover advantage**, no matter who moves first.

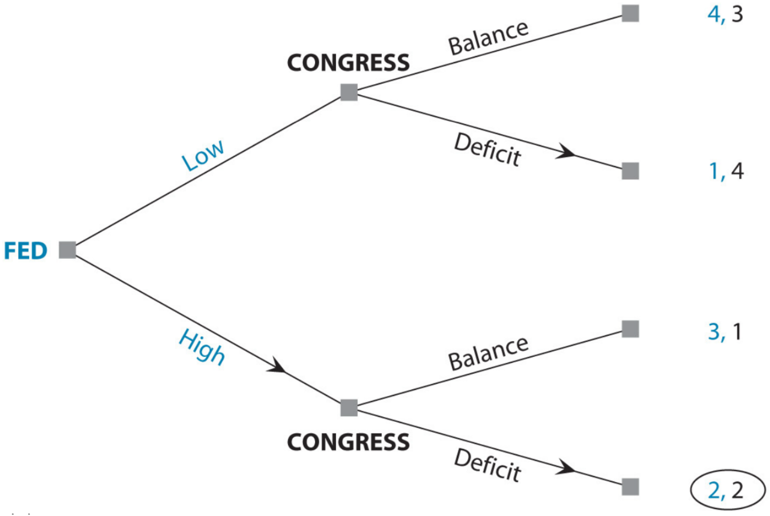
Games with Aligned Move Advantage

There are games where the order of moves can make both players better or worse off, depending on who goes first.

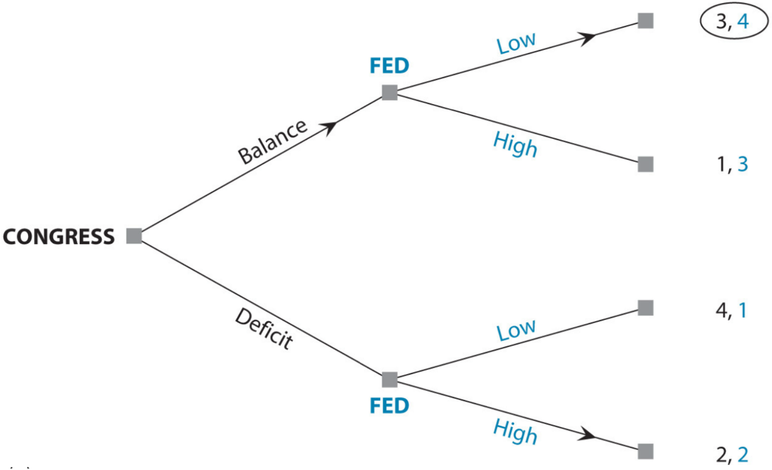
In this example, **Congress** chooses a row and the **Federal Reserve** chooses a column.

	Low interest rates	High interest rates
Budget balance	(3, 4)	(1, 3)
Budget deficit	(4, 1)	(2, 2)

Games with Aligned Move Advantage



Games with Aligned Move Advantage



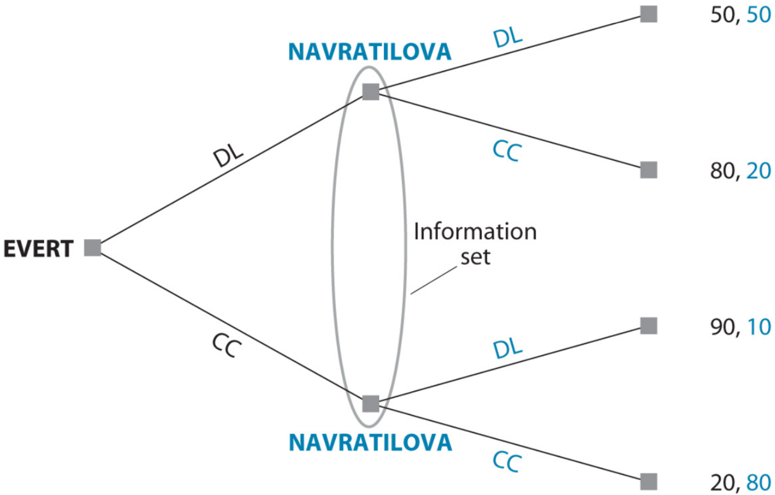
Simultaneous-Move Games and Game Trees

Simultaneous-move games can be represented using game trees. Unlike sequential games, players make their decisions **without observing the other player's move**.

- **Information Set:** A set of decision nodes between which a player cannot distinguish when making a choice. It reflects what the player knows at the time of decision-making.
- **Imperfect Information:** Occurs when an information set contains **more than one decision node**, meaning the player does not know the exact history of the game when choosing an action.

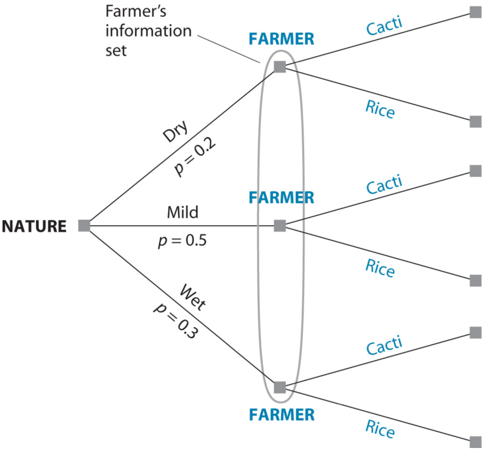
In simultaneous-move games, this lack of knowledge is captured by connecting decision nodes in the tree, visually indicating that the player makes a decision under uncertainty.

Simultaneous-Move Game Shown as a Game Tree



Simultaneous-Move Game Shown as a Game Tree

Game trees can also represent **external uncertainty**, such as weather or a random event. This is often modeled using an initial move by "Nature" with known probabilities.



Sequential-Move Games Using Matrices

Let's Analyze the Sequential-Move Game of Fiscal and Monetary Policy in a Normal Form

	L if B, H if D	H if B, L if D	Low always	High always
Balance	3,4	1,3	3,4	1,3
Deficit	2,2	4,1	4,1	2,2

Why the Matrix Form Has More NE

When we analyze the game using the matrix (strategic) form, we find two NE.

However, when we solve the sequential (extensive-form) version of the same game, we find a unique rollback equilibrium.

This discrepancy occurs because:

- The matrix form does not account for the timing of moves or what happens off the equilibrium path.
- Some Nash equilibria in the matrix rely on strategies that are **not credible** in subgames that are never reached.
- Rollback requires players to act optimally in **every subgame**, even off the equilibrium path.

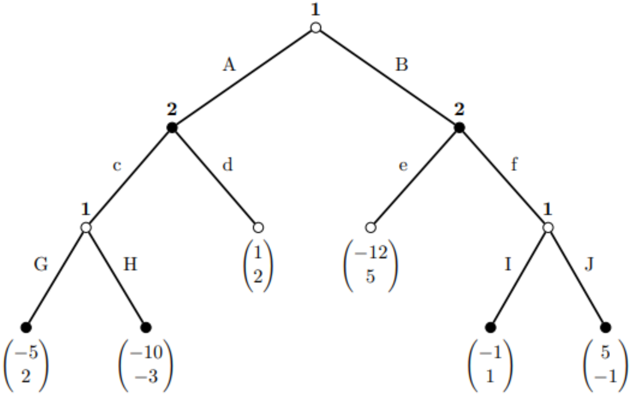
Rollback equilibrium refines NE by ruling out non-credible threats or promises.

Subgames

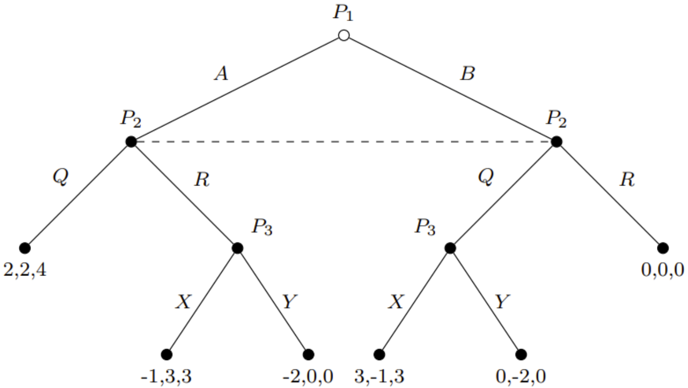
A **subgame** is any part of the game that:

- Begins at a single decision node that forms a **complete information set** — that is, the player knows exactly where they are in the game.
- Includes all future decision nodes and outcomes that follow from that point onward.
- Does not "cut through" any information set — it must preserve the structure of the game.

How many subgames are there in the following game?



How many subgames are there in the following game?



Subgame Perfect Equilibrium (SPE)

Subgame Perfect Equilibrium is a refinement of Nash Equilibrium that requires players to play optimally in **every subgame** of the game.

- SPE ensures that strategies are credible not just on the equilibrium path, but also in all possible subgames.
- It generalizes the idea of rollback equilibrium, which applies only to games of perfect information.
- SPE is particularly useful for games with **imperfect information**, where players may not know exactly which node they're in.

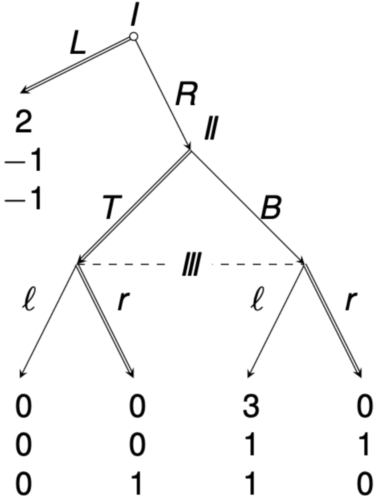
In any extensive-form game, an SPE is a strategy profile that represents a Nash Equilibrium in every subgame.

Example

Do the game tree for the initial game of these slides.

Example

How many subgames does the following tree have?
What is the SPE?



Example

The following game is played twice, which means the player should choose a strategy for $t = 1$ and $t = 2$. The total payoff is the sum of the two payoffs.

	L	C	R
T	$(4, 6)$	$(0, 0)$	$(9, 0)$
M	$(0, 0)$	$(6, 4)$	$(0, 0)$
B	$(0, 0)$	$(0, 0)$	$(8, 8)$

Analyze why the following strategy profile is a SPE:

- $s_1^1 = B, \quad s_1^2(s_1^1, s_2^1) = \begin{cases} M, \text{ if } s_1^1 = B \\ T, \text{ if } s_1^1 \neq B \end{cases}$
- $s_2^1 = R, \quad s_2^2(s_1^1, s_2^1) = \begin{cases} C, \text{ if } s_1^1 = B \\ L, \text{ if } s_1^1 \neq B \end{cases}$

Augmented PD

	E	S	P
E	$(2,2)$	$(-1,3)$	$(-1,-1)$
S	$(3,-1)$	$(0,0)$	$(-1,-1)$
P	$(-1,-1)$	$(-1,-1)$	$(-2,-2)$

Analyze what the following strategy is a NE, but not a SPE:

- $s_1^1 = E$, $s_1^2(s_1^1, s_2^1) = \begin{cases} S, & \text{if } s_2^1 = E \\ P, & \text{if } s_2^1 \neq E \end{cases}$
- $s_2^1 = E$, $s_2^2(s_1^1, s_2^1) = \begin{cases} S, & \text{if } s_1^1 = E \\ P, & \text{if } s_1^1 \neq E \end{cases}$