# Repeated Games

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# Motivation

- **One-shot dilemmas often end badly:**
  In the classic Prisoners' Dilemma, both players defect—even though mutual cooperation would be better.
- **But what if the game is played more than once?**
  Real-life interactions (e.g., between businesses, co-workers, or neighbors) are rarely one-time events.
- **Repetition opens the door to cooperation:**
  - Players can *build trust* over time.
  - They can *punish defection* in future rounds.
  - They can use *promises or threats* to influence behavior.
- **Key idea:** The future matters.
  Cooperation today may be rewarded tomorrow—or betrayal punished.

# The Standard Prisoners' Dilemma

| | **Wife** | |
|---|---|---|
| **Husband** | **Confess (Defect)** | **Deny (Cooperate)** |
| **Confess (Defect)** | (-10, -10) | (-1, -25) |
| **Deny (Cooperate)** | (-25, -1) | (-3, -3) |

- ✓ **Dominant strategy:** Confess (Defect), for both players.
- ✓ **Equilibrium:** (Confess, Confess) $\rightarrow$ each gets 10 years.
- ✓ **Mutual cooperation (Deny, Deny)** is better (3 years each), but unstable.

# Why Repetition Matters

- One-shot Prisoners' Dilemma leads to defection and a worse outcome for both players.
- But in many real-world settings, interactions are repeated over time.
- **Key idea:** Fear of losing future cooperation can sustain cooperation today.
- If the value of long-term cooperation is high enough, players will avoid short-term gains from defection.
- No need for external enforcement—cooperation can be self-sustaining.

Repetition transforms the logic of the game.

# Example: Restaurant Pricing Game

| | Yvonne's Bistro | |
|---|---|---|
| **Xavier's Tapas** | \$20 (Defect) | \$26 (Cooperate) |
| \$20 (Defect) | (288, 288) | (360, 216) |
| \$26 (Cooperate) | (216, 360) | (324, 324) |

- ✓ Collusive outcome (Cooperate, Cooperate): profits of **324** each.
- ✓ But (Defect, Defect) is the Nash equilibrium: both earn only **288**.
- ✓ If one defects (e.g. Xavier chooses \$20), he earns **360** in that round.

# Incentives in Repeated Play

**Scenario:** Xavier considers defecting once (price \$20) while Yvonne cooperates.

$\rightarrow$ **One-time gain:** 360 – 324 = +36 (i.e., \$3,600).

$\rightarrow$ **Future loss:** If cooperation breaks down:

- Xavier now earns 288 instead of 324.
- Loss = -36 per future month.

$$\text{Total future loss} > \text{Short-term gain} \Rightarrow 36 \times t > 36 \Rightarrow h > 1$$

Which means tomorrow having at least 2 periods (given $> 1$). Hence, if the relationship lasts at least 3 months (today and 2 periods of the future), it's better to stay cooperative.

The shadow of the future can discipline short-term temptation. Nonetheless, things are not that simple.

# Finite Repetition and the End-Game Effect

What if the game lasts exactly 3 months? We apply backward induction

$\rightarrow$ Month 3: No future punishment possible $\rightarrow$ best to defect.

$\rightarrow$ Month 2: Knowing both will defect in Month 3 $\rightarrow$ defect now.

$\rightarrow$ Month 1: Anticipating defection in Months 2 and 3 $\rightarrow$ defect from the start.

- Cooperation unravels backward: defection in every round.
- Even if repeated, a known end makes cooperation unsustainable.
- This is called the <u>end-game effect</u>.

In theory, finite games leads to defection. In practice, people tend to cooperate.

# Why Do We Discount the Future?

- ⋆ In repeated games, we compare future gains and losses to present ones.
- ⋆ But future payoffs are often **worth less** than present payoffs. Why?

  - *Impatience:* People tend to prefer benefits now rather than later.
  - *Opportunity cost of savings:* Money today can be invested to earn returns.

**Notation:**

- ∘ $r =$ **discount rate** (e.g., interest rate, rate of return)
- ∘ $\delta = \frac{1}{1+r} =$ **discount factor**. We will use $\delta$ to convert future payoffs into present utility.

# Infinite Repetition and Cooperation

- When the game has no known end, future consequences matter.
- Players can now use contingent strategies, where their choice depends on past behavior.
- A key class: **trigger strategies**, which punish defection to sustain cooperation.
- Two well-known trigger strategies:
  - **Grim strategy**: Cooperate until opponent defects, then defect forever.
  - **Tit-for-tat (TFT)**: Mirror your opponent's previous move.

With enough future at stake, cooperation becomes rational.

# Is It Worth Defecting Once Against Tit-for-Tat?

- Defecting once gives Xavier a one-time gain of 36.
- But Yvonne (playing TFT) punishes by defecting next month $\rightarrow$ Xavier loses 108.
- Xavier compares:

$$\text{Gain: } 36 \quad \text{vs.} \quad \text{Loss: } 108\delta$$

- Defection is profitable only if:

$$36 > 108\delta \Rightarrow \delta < \frac{1}{3}$$

- So, Xavier needs to be extremely impatient to have incentives to deviate.

# Defecting Forever: Worth It?

- Xavier defects and continues to do so, getting:

Gain in Month 1: 36

- He loses 36 every month after that due to Yvonne's grim retaliation.
- Present value of losses (infinite sum):

$$\sum_{n=1}^{\infty} 36\delta^n = \frac{36\delta}{1-\delta}$$

- Compare:

$$\text{Defecting is worth it if } 36 > \frac{36\delta}{1-\delta} \Rightarrow \delta < \frac{1}{2}$$

# Games of Unknown Length

- In many real-world repeated games, players don't know how long the interaction will last.
- Suppose the game continues to the next round with probability $p$.
- Future payoffs are now discounted by both:
  - the time discount factor $\delta = \frac{1}{1+r}$
  - and the probability of continuation $p$

$$\text{Effective discount factor: } p\delta$$

We just multiply the probability of continuation. The lower $p$, the less players care about the future $\rightarrow$ cooperation becomes harder to sustain.

# Evaluating Infinite Discounted Payoffs

A player who receives a constant payoff every period values the future less than the present.

For instance, suppose a player gets 6 every period forever. With discount factor $\delta$, the total value is:

$$6 + 6\delta + 6\delta^2 + 6\delta^3 + \cdots$$

This is a geometric series:

$$\text{Total payoff} = \frac{6}{1-\delta}$$

**More generally:** If the per-period payoff is $x$, then the value of receiving $x$ forever is:

$$\frac{x}{1-\delta}$$

# Exercise: Infinitely Repeated Prisoner's Dilemma

| | **Player 2** | |
|---|---|---|
| **Player 1** | **Cooperate (C)** | **Defect (D)** |
| **Cooperate (C)** | (3, 3) | (0, 5) |
| **Defect (D)** | (5, 0) | (1, 1) |

✓ Grim Trigger Strategy:
  - Play (C,C) in period 1
  - Continue playing (C,C) if no defection has occurred
  - If any deviation occurs $\rightarrow$ play (D,D) forever

✓ Is this a subgame perfect equilibrium (SPE)?

✓ Depends on $\delta$.

# Incentive to Cooperate in the Infinitely Repeated PD

| | **Player 2** | |
|---|---|---|
| **Player 1** | **Cooperate (C)** | **Defect (D)** |
| **Cooperate (C)** | (3, 3) | (0, 5) |
| **Defect (D)** | (5, 0) | (1, 1) |

**If both follow grim trigger:**

$$\text{Payoff from cooperating} = \frac{3}{1-\delta}$$

**If player 1 deviates once:**

$$\text{Payoff from defecting} = 5 + \frac{1 \cdot \delta}{1-\delta}$$

# continuation

**No incentive to deviate** if:

$$\frac{3}{1-\delta} \geq 5 + \frac{\delta}{1-\delta} \Rightarrow 3 \geq 5(1-\delta) + \delta \Rightarrow \delta \geq \frac{2}{5}$$

*Conclusion: Grim trigger is an SPE if and only if* $\delta \geq \frac{2}{5}$.

# Example: Alternating NE Strategy (from Full Matrix)

| | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| **a** | 9, 9 | 2, 4 | 1, 11 | 3, 0 |
| **b** | 4, 2 | 4, 4 | 2, 2 | 1, 1 |
| **c** | 11, 1 | 2, 2 | -1, -1 | 5, 3 |
| **d** | 0, 3 | 1, 1 | 3, 5 | 0, 0 |

# Grim Trigger to (b,b)

**Strategy:**

- Play $(a, a)$ in period 1.
- In any future period:
  - If $(a, a)$ was always played in the past, continue playing $(a, a)$.
  - Otherwise, switch permanently to $(b, b)$.

**SPE condition:** This is an SPE if $\delta \geq \frac{2}{7}$.

# Trying a Harsher Punishment

**New idea:** What if we punish more harshly?

- In period 1, play $(a, a)$.
- In any later period:
  - If $(a, a)$ was always played, continue with $(a, a)$.
  - If any deviation is observed, switch forever to $(d, d)$ or $(c, c)$.

*Student prompt:* "Can this help support $(a, a)$ even when $\delta < 2/7$?"

# What's the Problem with (d,d)?

**Suppose:** A deviation happens in period 1.

- Then punishment $(d, d)$ starts in period 2.
- But... is that punishment credible?

**Payoffs:**

- Follow: $0 + 0 + 0 + \cdots = 0$
- Deviation: $5 + 0 + 0 + \cdots = 5$

**Conclusion:** Player prefers to deviate!

# Why Isn't (d,d) a Valid Punishment?

**Key point:** (d,d) is **not** a Nash Equilibrium of the stage game.

- Grim-trigger to a non-NE doesn't work — players won't stick to it.
- The punishment itself must be an equilibrium path in the subgame.

*Lesson:* You can't enforce (a,a) forever by threatening something players won't actually follow through on!

# Tailoring the Punishment to the Deviator

**Smarter strategy:**

- ✓ In period 1: play $(a, a)$.
- ✓ In later periods:
  - Play $(a, a)$ if always played in the past.
  - If first deviation was $(a, c)$, punish with $(c, d)$.
  - Otherwise, punish with $(d, c)$.

**Result:** This is an SPE if $\delta \geq \frac{1}{4}$

*We have improved the bound!*

# An SPE with No Discounting Requirement

**Strategy:**

- In every odd period: play $(c, d)$
- In every even period: play $(d, c)$

**Why does this work?**

- $(c, d)$ and $(d, c)$ are both NE of the stage game.
- No incentive to deviate — players best respond in each round.
- Future behavior doesn't depend on history — no punishment required.

**Conclusion:** This is an SPE **for any** $\delta$!